Nonparametric tests of change-points with tapered data

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Abstract

In this work we build two families of nonparametric tests using tapered data for the off-line detection of change-points in the spectral characteristics of a stationary Gaussian process. This is done using the Kolmogorov-Smirnov's Statistics based on integrated tapered periodograms.

The convergence is obtained under the null hypothesis by means of a double indexed (frequency - time) process together with some extensions of Dirichlet and Fejer kernels. Consistency is proved using these statistics under the alternative.

Then using numerical simulations, we observe that the use of tapered data significantly improves the properties of the test, especially in the case of small samples.

KEY WORDS: Nonparametric, Tests, Change-Point, Periodogram, Tapered Data, Stationary Process.

Mathematics subject classifications (1991): 62G10, 62G20, 62M10, 62M15.

1 Introduction

The problem of detecting a change-point in the properties of a process has been extensively studied, see for a general survey the books [1], [2] and [7] or more recently the works of [21] or [28].

The change point problem can be formulated in two different ways: a sequential problem called "on-line" and another one a posteriory called "off-line". We consider here this last case where one has to decide between homogeneity and change-point after observing a given set of random data.

Different kinds of changes can affect a stochastic process: changes in the mean, in the covariance structure, etc. In the present work, our aim is to detect changes in the spectrum of a strictly stationary time series, while assuming no change in the mean. As the distributions are a priori unknown, we restrict ourselves to a nonparametric framework and we build two families of nonparametric test-statistics for change-point detection using estimates of the spectral measure.

Several authors have investigated such problems of detecting change in the spectral distribution function in an off-line case. We can quote for instance [31] for Gaussian processes, [19], [20], [21] for linear processes and [27] for multidimensional Gaussian processes.

On the other hand, it is well known that, for estimating the spectral measure of a stationary process, the use of the periodogram requires a large number of data. To bypass this problem of sample sizes, Dahlhaus has shown, in a series of papers [12] [14] [13] [15] that the use of tapered data improves spectral estimation: The increase in the asymptotic variance is balanced by a reduction of the bias which leads to better results for small samples sizes. This is the old remedy to reduce leakage effects pointed out by [36] or more recently in the papers of [38], [37], and [26].

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In this paper, we extend to tapered data the results on change point detection [31]. For this, following Dahlhaus, we introduce tapers, and then build two families of test-statistics which are related to the Kolmogorov-Smirnov test statistics. Assuming that the process is Gaussian, we show the asymptotic normality of a double indexed (frequency-time) process. This result allows us to precise the distribution of our tests under the null hypothesis where no change occurs. By numerical simulations, we show that using an appropriate taper depending on the sample size, improves significantly the detection for small sample size.

Section 2 contains the assumptions, notations and the definition of the auxiliary process Z_T on which the statistical studies relied. We prove that the test statistics derived from Z_T converge under the null hypothesis to a known distribution which only depends on the chosen taper. Therefore, it is possible to tabulate this limiting distribution according to the taper, and obtain in practice the associated tests. The result relies on a central limit theorem for Z_T stated in Theorem 2. Section 3 contains some preliminary results on the kernels used here. The functional central limit theorem for Z_T is proved in Section 4. Section 5 is devoted to the study of the asymptotic distributions of our statistics under the null hypothesis. The main point is to show that the two statistics have the same asymptotic distribution (Theorem 3). In Section 6, we study the asymptotic confidence region (Corollary 1.1) and the asymptotic consistency of the tests. Finally, we give some numerical results obtained with simulations. They show that, for small sample sizes ($T \sim 50$), the use of tapers improves the performance. Indeed, for the same number of false alarms, there are twice more true alarms detected using of taper. This effect tends to disappear as T increases.

2 Statement of the results

2.1 Notations and assumptions

Let $\mathcal{G}$ denote the set of stationary real centered Gaussian processes with absolutely continuous spectral measure with density f *w.r.t.* the normalized Lebesgue measure on the torus $\mathbb{T} = [-\pi, +\pi[$, $d\boldsymbol{\lambda}(\alpha) = d\alpha/2\pi$, which satisfy

- (A-i) $F(\lambda) = \int_0^\lambda f(\alpha)d\boldsymbol{\lambda}(\alpha)$ is an increasing function on $[0, \pi]$,
- (A-ii) $G_2 = \int_0^\pi f^2(\alpha)d\boldsymbol{\lambda}(\alpha) < +\infty$.

We test the following hypotheses

$\mathcal{H}_0$: "$(X_j)_{j=1\dots T}$ is the restriction to $\{1, \dots, T\}$ of a process belonging to $\mathcal{G}$".

against

$\mathcal{H}_1$: "there exists k_0, $1 < k_0 < T$ such that $X_j = \widetilde{X}_j^1$ if $j \leq k_0$ and $X_j = \widetilde{X}_j^2$ if $j > k_0$ where $\widetilde{X}^1$ and $\widetilde{X}^2$ are both in $\mathcal{G}$ and have two unknown different spectral measures F_1 and F_2 (F_i denotes the spectral measure of $\widetilde{X}^i$, $i = 1, 2$)".

Let h be a non negative C^1-function defined on $[0, 1]$: the taper. For $k = 0, 1, \dots, T$ and $s \in [0, 1]$, let us define

$$H_p^k(\alpha) = \sum_{j=1}^{k} h^p(\frac{j}{T})e^{-ij\alpha}, \quad H_p^k = H_p^k(0), \quad H_p^0 = 0, \quad H_p(s) = \int_0^s h^p(u)du, \quad H_p = H_p(1). \tag{1}$$

Consider now the tapered periodograms $I_k(\alpha)$ (respectively $\check{I}_{T-k}(\alpha)$ built on the first k data (resp. on the last $T-k$ data), as in [12]

$$d_k(\alpha) = \sum_{j=1}^{k} X_j h(\frac{j}{T})e^{-ij\alpha}, \quad I_k(\alpha) = \frac{|d_k(\alpha)|^2}{H_2^k}, \quad \check{I}_{T-k}(\alpha) = \frac{|d_T(\alpha) - d_k(\alpha)|^2}{H_2^T - H_2^k} \tag{2}$$

with $I_0(\alpha) = \check{I}_T(\alpha) = 0$. The associated estimates of the spectral distribution function are respectively

$$F_k(\lambda) = \int_0^\lambda I_k(\alpha) d\boldsymbol{\lambda}(\alpha), \quad \check{F}_{T-k}(\lambda) = \int_0^\lambda \check{I}_{T-k}(\alpha) d\boldsymbol{\lambda}(\alpha). \tag{3}$$

The asymptotic properties of F_T have been studied, in the non tapered case (h identically equal to 1), in [25], [29], and in the tapered case by [12] [14] [15]. In particular, Dahlhaus has first pointed out the interest of data tapering by numerical simulations results and then, through his theory of "high-resolution", [15] proved it. For a spectral density having high peaks the estimates based on tapered data behave better (especially around the peaks). Indeed, for small T, this is because their bias is reduced. For large T, they are uniformly more efficient with respect to the supremum of the integrated mean square error; this supremum is taken on a class of densities which contain the ARMA processes, whose roots belong to the disk with radius $1 - 1/T$.

Denote by $[a]$ the integer part of a and define

$$Y_T^1(\lambda, s) = \sqrt{T} \frac{H_2^{[sT]}}{H_2^T} \left(F_{[sT]}(\lambda) - F_T(\lambda) \right), \tag{4}$$

$$Y_T^2(\lambda, s) = \sqrt{T} \frac{H_2^{[sT]}}{H_2^T} \frac{H_2^T - H_2^{[sT]}}{H_2^T} \left(F_{[sT]}(\lambda) - \check{F}_{T-[sT]}(\lambda) \right). \tag{5}$$

The associated tests statistics are

$$S_T^i(h) = \sup_{\lambda \in [0,\pi]} \sup_{s \in [0,1]} \left| Y_T^i(\lambda, s) \right|, \qquad i = 1, 2 \tag{6}$$

and the critical regions are, using the notations of (A-ii) and (1)

$$R_T^j = \left\{ S_T^j(h) > \frac{c}{H_2} \sqrt{G_2 H_4} \right\}, \quad j = 1, 2. \tag{7}$$

The properties of the statistic $S_T^2(h)$ under the null hypothesis $\mathcal{H}_0$ in the non tapered case (i.e. $h \equiv 1$) have been studied by [31]. In the non tapered case, [20] have studied tests for linear (non necessarily Gaussian) processes using analogous test-statistics.

2.2 Main Results

Our first aim is to obtain the asymptotic distributions of the above statistics in order to get the asymptotic level of our test.

Let us stress upon the fact that the critical regions are not free. The unknown parameter G_2 defined in $(A - ii)$ remains in the test. Hence, this value must be known or substituted by a consistent estimate $\widetilde{G}_2$ in order to perform the test. For example G_2 can be replaced by

$$\widetilde{G}_2 = \frac{1}{2} \sum_{|u| < \sqrt{T}} \left(\frac{1}{T} \sum_{j=1}^{T-|u|} X_j X_{j+u} \right)^2. \tag{8}$$

Let $\mathcal{D} = \mathcal{D}([0, \pi] \times [0, 1], \mathbf{R})$ be the space of real functions on $[0, \pi] \times [0, 1]$ which are right-continuous with left-hand limits, endowed with the Skorohod topology. Let $\mathcal{C} = \mathcal{C}([0, \pi] \times [0, 1], \mathbf{R})$ be the space of continuous real functions on $[0, \pi] \times [0, 1]$, with the uniform convergence topology. Clearly processes Y_T^1 and Y_T^2 are $\mathcal{D}$-valued. The following holds

Theorem 1 *Assume that $(X_j)_{j\in\mathbf{N}}$ belongs to $\mathcal{G}$. Then, as $T\to+\infty$, under the null hypothesis $\mathcal{H}_0$, the two sequences (Y_T^1) and (Y_T^2) converge in distribution in $\mathcal{D}$ to the centered Gaussian process*

$$\frac{1}{H_2}\left(Z(\lambda,s)-\frac{H_2(s)}{H_2}Z(\lambda,1)\right)_{(\lambda,s)\in[0,\pi]\times[0,1]} \tag{9}$$

where H_2 and $H_2(.)$ are defined in (1) and where $Z(.,.)$ is the continuous Gaussian process with mean zero and covariance function

$$\mathbf{E}\,Z(\lambda,s)Z(\lambda',s')=\int_0^{s\wedge s'}h^4(u)du.\int_0^{\lambda\wedge\lambda'}f^2(\alpha)d\boldsymbol{\lambda}(\alpha). \tag{10}$$

The proof of Theorem 1 is given in Section 4.

Now, we can study the levels of the tests associated with the critical regions R_T^j defined in (7).

Corollary 1.1 *Assume that $(X_j)_{j\in\mathbf{N}}$ belongs to $\mathcal{G}$. Then, as $T\to+\infty$,*

$$\lim_{T\to+\infty}\mathbf{P}_{\mathcal{H}_0}(R_T^j)=\mathbf{P}(\sup_{u\in[0,1]}\sup_{s\in[0,1]}\left|B(u,s)-\frac{H_2[H_4^{-1}(s.H_4)]}{H_2}B(u,1)\right|>c)$$

where $H_4^{-1}(.)$ denotes the inverse (or pseudo-inverse) function of $H_4(.)$ and B is the standard bidimensional Brownian motion on $\mathbf{R}^2$ centered with covariance function $\mathbf{E}\,B(u,s)B(u',s')=(u\wedge u').(s\wedge s')$.

The proof in given in Section 6.
Therefore, for T large enough, we obtain approximations for the levels which only depend on the taper h. To obtain this result, we introduce, as in [31] [20], the process Z_T with values in $\mathcal{D}$ defined by

$$Z_T(\lambda,s)=\frac{H_2^{[sT]}}{\sqrt{T}}(F_{[sT]}(\lambda)-F(\lambda)). \tag{11}$$

Then, the following holds for Y_T^1 and Y_T^2,

$$Y_T^1(\lambda,s) = \frac{T}{H_2^T}\left(Z_T(\lambda,s)-\frac{H_2^{[sT]}}{H_2^T}Z_T(\lambda,1)\right) \tag{12}$$

$$Y_T^2(\lambda,s) = Y_T^1(\lambda,s)+\frac{TH_2^{[sT]}}{(H_2^T)^2}\Omega_T(\lambda,s) \tag{13}$$

where

$$\Omega_T(\lambda,s)=\frac{2}{\sqrt{T}}\mathbf{Re}\int_0^\lambda\left(\sum_{j=1}^{[sT]}h(\frac{j}{T})X_je^{-ij\alpha}\sum_{j=[sT]+1}^{T}h(\frac{j}{T})X_je^{ij\alpha}\right)d\boldsymbol{\lambda}(\alpha). \tag{14}$$

All the results quoted above rely on the next theorem which states, under the null hypothesis $\mathcal{H}_0$, the weak convergence of the process $Z_T(.,.)$.

Theorem 2 *Assume that $(X_j)_{j\in\mathbf{N}}$ belongs to $\mathcal{G}$. Then $(Z_T)_{T\in\mathbf{N}}$ converges in distribution to the centered Gaussian process Z of $\mathcal{C}([0,\pi]\times[0,1],\mathbf{R})$ with covariance function given in (10).*

The proof of Theorem 2, is given in Section 4. Some preliminary results are useful for obtaining the limiting covariances together with the tightness of the sequence $Z_T(.,.)$.

3 Preliminary results

This section is devoted to studying the kernels that appear in the proof of the functional central limit theorem.

Let us define, for p and j arbitrary integers and for $\alpha \in [-\pi, \pi]$, the functions $\widetilde{L}_j^p$

$$\widetilde{L}_j^p(\alpha) = \begin{cases} j\frac{\pi}{2}, & \text{if } |\alpha| \leq \frac{2}{j}, \\ \pi(p^{-p})\frac{\log^p(e^p j|\alpha|/2)}{|\alpha|}, & \text{if } \frac{2}{j} \leq |\alpha| \leq \pi. \end{cases}$$

For $\alpha \in \mathbf{R}$, denote by L_j^p the 2π-periodic extension of $\widetilde{L}_j^p$.

Property 1 *The function $L_j^p(\alpha)$ has the following properties*

1. *For all p and j, $\alpha \to L_j^p(\alpha)$ is continuous, odd, and, for α in $[0, \pi]$, non increasing. It is an increasing function w.r.t. j and p.*
2. *For all p, q, $n > 1$ and j, there is a constant K, which does not depend on j such that*

$$(a). \quad \int_{\mathbf{T}} L_j^p(\alpha) d\boldsymbol{\lambda}(\alpha) \leq K(\log j)^{p+1}, \qquad (b). \quad \int_{\mathbf{T}} \left[L_T^p(\alpha)\right]^n d\boldsymbol{\lambda}(\alpha) \leq KT^{n-1}, \tag{15}$$

$$(c). \quad \int_{\mathbf{T}} L_j^p(\gamma + \alpha) L_j^q(\beta - \alpha) d\boldsymbol{\lambda}(\alpha) \leq K L_j^{p+q+1}(\beta + \gamma). \tag{16}$$

The proof is omitted here, since this result is similar to the one stated in [12, lemma 1 and 2]. The only difference lies in the definition of the L_T^p. A detailed proof can be found in [33].

Lemma 3.1 *Let $|\alpha| \leq \pi$, and consider two integers k_1, k_2, $k_1 < k_2$, set*

$$\left| \sum_{j=k_1+1}^{k_2} h(\frac{j}{T}) e^{-ij\alpha} \right| \leq C_h L_{k_2-k_1}^0(\alpha).$$

Proof Now, let us consider the Dirichlet kernels $\Delta_k(\alpha) = H_0^k(\alpha)$ (see (1)). By an Abel transformation, we obtain

$$\sum_{j=k_1+1}^{k_2} h(\frac{j}{T}) e^{-ij\alpha} = \left(h(\frac{k_2}{T}) \Delta_{k_2-k_1}(\alpha) - \sum_{j=k_1+1}^{k_2-1} \left[h(\frac{j+1}{T}) - h(\frac{j}{T}) \right] \Delta_{j-k_1}(\alpha) \right) e^{-ik_1\alpha}.$$

So,

$$\left| \sum_{j=k_1+1}^{k_2} h(\frac{j}{T}) e^{-ij\alpha} \right| \leq \left(h(\frac{k_2}{T}) + \sum_{j=k_1+1}^{k_2-1} \left| h(\frac{j+1}{T}) - h(\frac{j}{T}) \right| \right) \sup_{j=1,\ldots,k_2-k_1} |\Delta_j(\alpha)|.$$

As h is $\mathcal{C}^1$, the term between brackets is bounded by a constant C_h depending only on h. As the functions $L_j^p(\alpha)$ are increasing with j, we just have to prove that, for $|\alpha| \leq \pi$

$$|\Delta_j(\alpha)| = \left| \frac{\sin(j\alpha/2)}{\sin(\alpha/2)} \right| \leq L_j^0(\alpha).$$

Since $\sin x \leq x$, for $x \in [0, \pi]$ and $\frac{2}{\pi}x \leq \sin x$ for $x \in [0, \pi/2]$, then one easy checks that

- for $|\alpha| \le 2/j$,

$$|\Delta_j(\alpha)| \le \frac{j\frac{|\alpha|}{2}}{\frac{2}{\pi}\frac{|\alpha|}{2}} = \frac{\pi j}{2} = L_j^0(\alpha), \tag{17}$$

- for $2/j \le |\alpha| \le \pi$,

$$|\Delta_j(\alpha)| \le \frac{1}{\frac{2}{\pi}\frac{|\alpha|}{2}} = \frac{\pi}{|\alpha|} = L_j^0(\alpha). \tag{18}$$

This completes the proof of Lemma 3.1

Q.E.D.

Let us now introduce some kernels based on the taper h, which are approximate identities for convolution. Set $\underline{s} = \min(s_1, \ldots, s_m)$ and define on $\mathbf{T}^{m-1}$ the function $\Phi_T^{s_1,\ldots,s_m}$ by

$$\Phi_T^{s_1,\ldots,s_m}(\gamma_1, \ldots, \gamma_{m-1}) = \begin{cases} \dfrac{H_1^{[s_1T]}(\gamma_1)\ldots H_1^{[s_{m-1}T]}(\gamma_{m-1})H_1^{[s_mT]}(-\sum_{j=1}^{m-1}\gamma_j)}{H_m^{[\underline{s}T]}}, & \text{if } H_m^{[\underline{s}T]} \ne 0, \\ 1, & \text{otherwise.} \end{cases} \tag{19}$$

Lemma 3.2 *Let $m \in \mathbf{N}$, $s_1, \ldots, s_m$ in $[0,1]$ and h be $\mathcal{C}^1$ such that $H_m(\underline{s}) \ne 0$. Then, the sequence of functions $(\Phi_T^{s_1,\ldots,s_m})_{T=1}^{\infty}$ is an approximate identity for convolution.*

First (see [34, Chaper 6]), recall that a family $(\Phi_T)_{T\in\mathbf{N}}$ of Lebesgue integrable functions on $\mathbf{T}^{m-1}$ with values in $\mathbf{C}$ is called an approximate identity for convolution if, for $\gamma = (\gamma_1, \ldots, \gamma_{m-1})$ and $||\gamma|| = \sup_{i=1,\ldots,m-1} |\gamma_i|$,

$$(i) \quad \sup_{T\in\mathbf{N}} \int_{\mathbf{T}^{m-1}} |\Phi_T(\gamma)|\, d\boldsymbol{\lambda}(\gamma) < \infty, \quad (ii) \lim_{T\to\infty} \int_{\mathbf{T}^{m-1}} \Phi_T(\gamma) d\boldsymbol{\lambda}(\gamma) = 1,$$

$$(iii) \quad \lim_{T\to\infty} \int_{\mathbf{T}^{m-1}\setminus\{\gamma, ||\gamma||<\delta\}} |\Phi_T(\gamma)|\, d\boldsymbol{\lambda}(\gamma) = 0, \qquad \text{for all } \delta > 0.$$

Then, for all bounded continuous complex functions g on $\mathbf{T}^{m-1}$, $\lim_{T\to\infty} ||\Phi_T * g - g||_\infty = 0$.
Proof This result was shown by [12, Lemma 3] in the case $s_1 = \ldots = s_m = 1$. We follow the sketch of his proof in our case,that is to say that $(\Phi_T^{s_1,\ldots,s_m})_{T=1}^{\infty}$ satisfies the three assertions (i)-(iii). As the functions $L_k^0(\alpha)$ increase with k, by applying Lemma 3.1 to $k_1 = 0$ and $k_2 = [sT]$, we get

$$H_1^{[sT]}(\alpha) \le C_h\, L_{[sT]}^0(\alpha) \le C_h\, L_T^0(\alpha).$$

Therefore,

$$|\Phi_T^{s_1,\ldots,s_m}(\gamma_1, \ldots, \gamma_{m-1})| \le \frac{(C_h)^m}{H_m^{[\underline{s}T]}} L_T^0(\gamma_1) \ldots L_T^0(\gamma_{m-1}) L_T^0(-\sum_{j=1}^{m-1}\gamma_j). \tag{20}$$

Assertion (i) is obtained noting that the limit of the sequence $(\frac{H_m^{[\underline{s}T]}}{T})$ is $H_m(\underline{s}) \ne 0$ by assumption, and using inequality (16).

Set $h_s = h.\mathbf{1}_{[0,s]}$ and let us check (ii). One has

$$\int_{\mathbf{T}^{m-1}} \Phi_T^{s_1,\ldots,s_m}(\gamma) d\boldsymbol{\lambda}(\gamma) = \frac{1}{H_m^{[\underline{s}T]}} \sum_{j_1=1}^{T} \cdots \sum_{j_m=1}^{T} h_{s_m}(\frac{j_m}{T}) \prod_{l=1}^{m-1} h_{s_l}(\frac{j_l}{T}) \int_{\mathbf{T}} e^{i(j_l-j_m)\gamma_l} d\boldsymbol{\lambda}(\gamma_l)$$

$$= \frac{1}{H_m^{[\underline{s}T]}} \sum_{j=1}^{T} \left(h_{s_1}(\frac{j}{T}) \ldots h_{s_m}(\frac{j}{T}) \right) = \frac{1}{H_m^{[\underline{s}T]}} \sum_{j=1}^{T} \left[h_{\underline{s}}(\frac{j}{T}) \right]^m = \frac{1}{H_m^{[\underline{s}T]}} \sum_{j=1}^{[\underline{s}T]} \left[h(\frac{j}{T}) \right]^m = 1.$$

Fix $\delta > 0$ and T_0 such that $\delta \geq \frac{\pi}{T_0}$. Now, using that $\{||\gamma|| \geq \delta\} = \cup_{j_0=1}^{m-1}\{|\gamma_{j_0}| \geq \delta\}$ and noticing that $|\gamma_{j_0}| \geq \delta \geq \frac{2}{T}$ implies $L_T^0(\gamma_{j_0}) = \frac{2}{\gamma_{j_0}} \leq \frac{2}{\delta}$, inequality (20) leads to

$$\int\limits_{\mathbf{T}^{m-1}\setminus\{\gamma,||\gamma||<\delta\}} |\Phi_T(\gamma_1,\ldots,\gamma_{m-1})| \bigotimes_{j=1}^{m-1} d\boldsymbol{\lambda}(\gamma_j)$$
$$\leq \frac{\pi(C_h)^m}{\delta H_m^{[sT]}} \sum_{j_0=1}^{m-1} \int_{\mathbf{T}^{m-2}} L_T^0(-\sum_{j=1}^{m-1}\gamma_j) \prod_{\substack{1 \leq j \leq m-1 \\ j \neq j_0}} L_T^0(\gamma_j) d\boldsymbol{\lambda}(\gamma_1)\ldots d\boldsymbol{\lambda}(\gamma_{m-1}).$$

Then, inequalities (15) give that this last term is an $O(\frac{\ln^{m-1} T}{T})$. We obtain assertion (iii) using again the assumption $H_m(\underline{s}) \neq 0$ to control $H_m^{[sT]}$. This completes the proof of Lemma 3.2

Q.E.D.

4 Functional central limit theorem

To obtain a functional limit theorem for the process Z_T defined in (11), we follow a classical scheme. We first investigate the limits of the finite dimensional distributions of Z_T and then we prove the tightness of Z_T.

4.1 Finite dimensional distributions

We first prove that Z_T is asymptotically uniformly centered, then exhibit its limiting covariance function and finally study its finite dimensional distributions.

Proposition 1 *Suppose (X_j) belongs to $\mathcal{G}$. Then, as $T \to \infty$*

$$\sup_{s\in[0,1],\lambda\in[0,\pi]} |\mathbf{E}\, Z_T(\lambda,s)| = o(1).$$

Proof We extend to tapered data inequality (1.5) of [25, p. 369]. If f is a function defined on the torus $\mathbf{T}$, denote by $\hat{f}(n)$ its n^{th} Fourier coefficient. The expectation of Z_T satisfies, using definitions (1), (2) and (11)

$$\mathbf{E}\, Z_T(\lambda,s) = \frac{H_2^{[sT]}}{\sqrt{T}} \int\limits_0^\lambda \left(\mathbf{E}\, I_{[sT]}(\alpha) - f(\alpha)\right) d\boldsymbol{\lambda}(\alpha) = \frac{H_2^{[sT]}}{\sqrt{T}} \sum_{n=-\infty}^{+\infty} \widehat{\varphi}(n) \left(\widehat{\mathbf{E}\, I_{[sT]}}(n) - \widehat{f}(n)\right),$$

with $\varphi = \mathbf{1}_{[0,\lambda]}$ and $\widehat{\varphi}(n) = (e^{-in\lambda}-1)/n$. Now, let $K_k(\alpha) = \frac{1}{k}\left|H_1^k(\alpha)\right|^2$ be the tapered generalization of the Fejer kernel. It follows that $\mathbf{E}\, I_k(\alpha) = K_k * f(\alpha)$, so that $\widehat{\mathbf{E}\, I_k}(n) = \widehat{K_k}(n).\widehat{f}(n)$. Hence, the Fourier coefficients of K_k are null for $n > k$, since they verify

$$\widehat{K_k}(n) = \frac{1}{H_2^k} \sum_{j=\sup(1,(1-n))}^{\inf(k,(k-n))} h(\frac{j}{T})h(\frac{j+n}{T}).$$

The expectation of Z_T splits into two terms

$$\mathbf{E}\, Z_T(\lambda,s) = \frac{H_2^{[sT]}}{\sqrt{T}} \left(\sum_{n=-[sT]}^{[sT]} \widehat{\varphi}(n)\widehat{f}(n)\left(\widehat{K_{[sT]}}(n) - 1\right) - \sum_{|n|>[sT]} \widehat{\varphi}(n)\widehat{f}(n)\right) = \frac{H_2^{[sT]}}{\sqrt{T}}\left(A_{[sT]} - B_{[sT]}\right) \tag{21}$$

with $A_{[sT]} = \sum\limits_{n=-[sT]}^{[sT]} \widehat{\varphi}(n)\widehat{f}(n)\left(\widehat{K_{[sT]}}(n) - 1\right)$ and $B_{[sT]} = \sum\limits_{|n|>[sT]} \widehat{\varphi}(n)\widehat{f}(n)$.

Using $|n\widehat{\varphi}(n)| \leq 2$ and the Schwarz inequality, we get $B^2_{[sT]} \leq 4 \sum\limits_{|n|>[sT]} \frac{1}{|n|^2} \sum\limits_{|n|>[sT]} \left|\widehat{f}(n)\right|^2$. Since $\sum\limits_{|n|>[sT]} \frac{1}{|n|^2} \leq 2[sT]^{-1}$, we get

$$\frac{H_2^{[sT]}}{\sqrt{T}} \left|B_{[sT]}\right| \leq ||h||^2 \left(\frac{[sT]}{T} \sum_{|n|>[sT]} |\widehat{f}(n)|^2 \right)^{1/2}. \tag{22}$$

Fix $\varepsilon > 0$, the behavior of $B_{[sT]}$ varies according to the position of s w.r.t. ε.

- If $0 \leq s < \varepsilon$, using $\frac{[sT]}{T} \leq \varepsilon$ and the Parseval equality yields $\frac{H_2^{[sT]}}{\sqrt{T}} \left|B_{[sT]}\right| \leq ||h||^2 \, \|f\|_2 \varepsilon^{1/2}$.
- If $\varepsilon \leq s \leq 1$, as $\frac{[sT]}{T} \leq 1$, $\frac{H_2^{[sT]}}{\sqrt{T}} \left|B_{[sT]}\right| \leq ||h||^2 \left(\sum_{|n|>[\varepsilon T]} |\widehat{f}(n)|^2\right)^{1/2}$.

Taking $\varepsilon = 1/\sqrt{T}$, we get that

$$\sup_{s\in[0,1]} \frac{H_2^{[sT]}}{\sqrt{T}} \left|B_{[sT]}\right| \leq ||h||^2 \sup \left(\frac{1}{\sqrt{T}}, \sum_{|n|>\sqrt{T}} |\widehat{f}(n)|^2 \right)^{1/2} = o(1). \tag{23}$$

To study $A_{[sT]}$, first consider

$$\widehat{\Delta_{[sT]}}(n) - 1 = \frac{1}{H_2^{[sT]}} \left(\sum_{j=\sup(1,(1-n))}^{\inf([sT],([sT]-n))} h(\frac{j}{T}) \left[h(\frac{j+n}{T}) - h(\frac{j}{T}) \right] - \sum_j^{(n)} h^2(\frac{j}{T}) \right) \tag{24}$$

where $\sum_j^{(n)}$ means $\sum_{j=[sT]-n+1}^{[sT]}$ if $n > 0$ and $\sum_{j=1}^{-n}$ if $n < 0$. This sum has at most n terms, so

$$\sum_j^{(n)} h^2(\frac{j}{T}) \leq |n| \, \|h\|^2.$$

Expanding h in Taylor series, leads to

$$H_2^{[sT]} \left| \widehat{\Delta_{[sT]}}(n) - 1 \right| \leq [sT] \|h\| \frac{|n|}{T} \|h'\| + |n| \|h\|^2 \leq C_h \, |n| \, . \tag{25}$$

Plugging this in $A_{[sT]}$ (see 21), we get

$$|\frac{H_2^{[sT]}}{\sqrt{T}} A_{[sT]}| \leq \frac{C_h}{\sqrt{T}} \sum_{|n|\leq[sT]} |n\widehat{\varphi}(n)\widehat{f}(n)| \leq C_h \frac{2}{\sqrt{T}} \sum_{|n|\leq[sT]} |\widehat{f}(n)| \leq C_h \frac{2}{\sqrt{T}} \sum_{|n|\leq T} |\widehat{f}(n)|. \tag{26}$$

As f is in L^2, $\sum_{|n|\leq T} |\widehat{f}(n)| = o(\sqrt{T})$, so the expression in the right hand-side of (26) goes to 0 as $T \to +\infty$ uniformly in $s \in [0,1]$. Joining this and (23) completes the proof of Proposition 1

Q.E.D.

Proposition 2 *If (X_j) is in $\mathcal{G}$, then, for all s_1 and s_2 in $[0,1]$ and all λ_1 and λ_2 in $[0,\pi]$,*

$$\lim_{T\to\infty} cov[Z_T(\lambda_1, s_1), Z_T(\lambda_2, s_2)] = \int_0^{\lambda_1\wedge\lambda_2} f^2(\alpha) d\boldsymbol{\lambda}(\alpha) \int_0^{s_1\wedge s_2} h^4(u) du.$$

Proof As $Z_T(.,.)$ is uniformly centered, it is enough to prove this result for the centered process $\widetilde{Z}_T$ defined by

$$\widetilde{Z}_T(\lambda, s) = Z_T(\lambda, s) - \mathbf{E}\, Z_T(\lambda, s) = \frac{1}{\sqrt{T}} \int_0^\lambda \left(|d_{[sT]}(\alpha)|^2 - \mathbf{E}\,|d_{[sT]}(\alpha)|^2\right) d\boldsymbol{\lambda}(\alpha) \tag{27}$$

where $d_k(\alpha)$ is defined in (2). Clearly,

$$cov(\widetilde{Z}_T(\lambda_1, s_1), \widetilde{Z}_T(\lambda_2, s_2)) = \int_0^{\lambda_1}\int_0^{\lambda_2} G(\alpha_1, \alpha_2, s_1, s_2) d\boldsymbol{\lambda}(\alpha_1) d\boldsymbol{\lambda}(\alpha_2) \tag{28}$$

where $G(\alpha_1, \alpha_2, s_1, s_2) = \mathbf{E}\left|d_{[s_1T]}(\alpha_1)\right|^2 \left|d_{[s_2T]}(\alpha_2)\right|^2 - \mathbf{E}\left|d_{[s_1T]}(\alpha_1)\right|^2 \mathbf{E}\left|d_{[s_2T]}(\alpha_2)\right|^2$. Let us study the term $G(\alpha_1, \alpha_2, s_1, s_2)$. Since $d_k(.)$ is a Gaussian process, setting $\xi_1 = d_{[s_1T]}(\alpha_1)$, $\xi_2 = d_{[s_1T]}(-\alpha_1)$, $\xi_3 = d_{[s_2T]}(\alpha_2)$ and $\xi_4 = d_{[s_2T]}(-\alpha_2)$, vector $(\xi_1, \xi_2, \xi_3, \xi_4)$ satisfies the Gaussian identity

$$\mathbf{E}\,\xi_1\xi_2\xi_3\xi_4 = \mathbf{E}\,\xi_1\xi_2\mathbf{E}\,\xi_3\xi_4 + \mathbf{E}\,\xi_1\xi_3\mathbf{E}\,\xi_2\xi_4 + \mathbf{E}\,\xi_1\xi_4\mathbf{E}\,\xi_2\xi_3. \tag{29}$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\,\xi_1\xi_3\mathbf{E}\,\xi_2\xi_4 \\
&= \left(\sum_{j_1=1}^{[s_1T]} h(\frac{j_1}{T}) e^{-ij_1\alpha_1} \sum_{j_3=1}^{[s_2T]} h(\frac{j_3}{T}) e^{ij_3\alpha_2} \int_{\mathbf{T}} e^{-i(j_1-j_3)\beta_1} f(\beta_1) d\boldsymbol{\lambda}(\beta_1)\right) \\
&\quad \left(\sum_{j_2=1}^{[s_1T]} h(\frac{j_2}{T}) e^{-iij_2\alpha_1} \sum_{j_4=1}^{[s_2T]} h(\frac{j_4}{T}) e^{ij_4\alpha_2} \int_{\mathbf{T}} e^{-i(j_2-j_4)\beta_2} f(\beta_2) d\boldsymbol{\lambda}(\beta_2)\right) \\
&= \int_{\mathbf{T}^2} d\boldsymbol{\lambda}(\beta_1) d\boldsymbol{\lambda}(\beta_2) f(\beta_1) f(\beta_2) \left[H_1^{[s_1T]}(\alpha_1+\beta_1) H_1^{[s_2T]}(-\alpha_2-\beta_1) H_1^{[s_1T]}(-\alpha_1+\beta_2) H_1^{[s_2T]}(\alpha_2-\beta_2)\right]
\end{aligned}$$

where $H_p^k(\alpha)$ is defined in (1). Substituting α_2 in $-\alpha_2$, we obtain the expression of $\mathbf{E}\,\xi_1\xi_4\mathbf{E}\,\xi_2\xi_3$. Now for $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, let us define

$$\begin{aligned}
\widetilde{g}(\gamma) \;&=\; \int_{\mathbf{T}} \mathbf{1}_{\lambda_1}(\alpha) \mathbf{1}_{\lambda_2}(\gamma_1+\gamma_2-\alpha) f(\gamma_1-\alpha) f(\gamma_3+\alpha) d\boldsymbol{\lambda}(\alpha) \\
&\quad + \int_{\mathbf{T}} \mathbf{1}_{\lambda_1}(\alpha) \mathbf{1}_{\lambda_2}(\alpha-\gamma_1-\gamma_2) f(\gamma_1-\alpha) f(\gamma_3+\alpha) d\boldsymbol{\lambda}(\alpha)
\end{aligned} \tag{30}$$

and

$$\Phi_T^{s_1,s_2,s_3,s_4}(\gamma) = \frac{H_1^{[s_1T]}(\gamma_1) H_1^{[s_2T]}(\gamma_2) H_1^{[s_3T]}(\gamma_3) H_1^{[s_4T]}(-\sum_{j=1}^{3}\gamma_j)}{H_4^{[s_1T]\wedge[s_2T]}}. \tag{31}$$

With these notations the covariances, defined in (28), may be expressed as

$$cov\left(\widetilde{Z}_T(\varphi_1, s_1), \widetilde{Z}_T(\varphi_2, s_2)\right) = \frac{H_4^{[s_1T]\wedge[s_2T]}}{T} \Phi_T^{s_1,s_1,s_2,s_2} * \widetilde{g}(0,0,0). \tag{32}$$

Noting that $\widetilde{g}(0,0,0) = \int_0^{\lambda_1\wedge\lambda_2} f^2(\alpha) d\boldsymbol{\lambda}(\alpha)$, and that $H_4^{[s_1T]\wedge[s_2T]}/T$ tends to $s_1 \wedge s_2$ as $T \to \infty$, we achieve the proof using Lemma 3.2.

Q.E.D.

Proposition 3 *Assume that (X_j) is in $\mathcal{G}$. Then, for all $\lambda_1, \ldots, \lambda_l$ in $[0, \pi]$, all $s_1, \ldots, s_m$ in $[0, 1]$, and all real numbers a_{j_1,j_2}, $j_1 = 1, \ldots, l$, $j_2 = 1, \ldots, m$, the random variable*

$$\sum_{j_1=1}^{l} \sum_{j_2=1}^{m} a_{j_1,j_2} \widetilde{Z}_T(\lambda_{j_1}, s_{j_2}) / \mathbf{Var} \left(\sum_{j_1=1}^{l} \sum_{j_2=1}^{m} a_{j_1,j_2} \widetilde{Z}_T(\lambda_{j_1}, s_{j_2}) \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Proof The asymptotic normality of the finite dimensional distributions is obtained using the same approach as Ibragimov in [25, Th. 3.1 pp.379-80]. In the quadratic decomposition proposed by Ibragimov

$$\sum_{j_1=1}^{l} \sum_{j_2=1}^{m} a_{j_1,j_2} \widetilde{Z}_T(\lambda_{j_1}, s_{j_2}) = (B_T \hat{X}_T, \hat{X}_T) - \mathbf{E} \, (B_T \hat{X}_T, \hat{X}_T)$$

where $\hat{X}_T$ is the column vector ${}^t(X_1, \ldots, X_T)$, only the $T \times T$ matrix B_T changes

$$B_T = \left(\frac{1}{\sqrt{T}} \sum_{j_1=1}^{l} \sum_{j_2=1}^{m} a_{j_1,j_2} \int_0^{\lambda_{j_1}} h(\frac{k_1}{T}) h(\frac{k_2}{T}) e^{i(k_2-k_1)\alpha} \mathbf{1}_{1 \le k_1, k_2 \le [s_{j_2} T]} d\boldsymbol{\lambda}(\alpha) \right)_{1 \le k_1, k_2 \le T}$$

So we only need to generalize the upper-bound for $||B_T||$. We have

$$\begin{aligned} ||B_T|| &= \frac{1}{\sqrt{T}} \sup_{||x||=1} \left| \sum_{j_1,j_2} a_{j_1,j_2} \sum_{k_1,k_2=1}^{T} x_{k_1} \overline{x_{k_2}} \int_0^{\lambda_{j_1}} h(\frac{k_1}{T}) h(\frac{k_2}{T}) e^{i(k_2-k_1)\alpha} d\boldsymbol{\lambda}(\alpha) \right| \\ &\le \frac{ml}{\sqrt{T}} \sup_{j_1,j_2} |a_{j_1,j_2}| \sup_{||x||=1} \int_{\mathbf{T}} \left| \sum_{k=1}^{T} x_k h(\frac{k}{T}) e^{-ik\alpha} \right|^2 d\boldsymbol{\lambda}(\alpha). \end{aligned}$$

This is the L_2-norm of the function having for Fourrier coefficients the finite sequence $(x_1 h(\frac{1}{T}), \ldots, x_T h(\frac{T}{T}))$. Hence, using the Parseval equality and the Schwarz inequality

$$\int_{\mathbf{T}} \left| \sum_{k=1}^{T} x_k h(\frac{k}{T}) e^{-ik\alpha} \right|^2 d\boldsymbol{\lambda}(\alpha) = x_1^2 h^2(\frac{1}{T}) + \ldots + x_T^2 h^2(\frac{T}{T}) \le ||h||_\infty^2 \, ||x||^2 \le ||h||_\infty^2 \, .$$

Thus,

$$||B_T|| \le \frac{ml}{\sqrt{T}} \sup_{j_1,j_2} |a_{j_1,j_2}| \, ||h||_\infty^2 \, . \tag{33}$$

So $||B_T||$ is always a $O(T^{-1/2})$ and we can apply the sketch of proof proposed by Ibragimov.

Q.E.D.

4.2 Tightness

To prove the tightness of the process $Z_T(.,.)$ we use as in [31] the *Csensov tightness criterion* (see [10]) for continuous processes:

A family of process $\{Y_T, T > 0\}$ of $\mathcal{C}([0, \pi] \times [0, 1], \mathbf{R})$ is tight if

1. *The family $\{Y_T(0, 0), T > 0\}$ is tight in $\mathbf{R}$,*
2. *The family $\{Y_T(0, .), T > 0\}$ is tight in $\mathcal{C}([0, 1], \mathbf{R})$,*
3. *The family $\{Y_T(., 0), T > 0\}$ is tight in $\mathcal{C}([0, \pi], \mathbf{R})$,*

4. *There exist constants $C > 0$, $\gamma_1 > 1$, $\gamma_2 > 0$ and a modulus of continuity $\tilde{\omega}$ defined on $[0, \pi]$ such that, for all $B = [\lambda_1, \lambda_2] \times [s_1, s_2]$ included in $[0, \pi] \times [0, 1]$,*

$$\mathbf{E} \left| \widehat{Y}_T(B) \right|^{\gamma_1} \leq C[(s_2 - s_1)\tilde{\omega}(\lambda_2 - \lambda_1)]^{\gamma_2}$$

where $\widehat{Y}_T(B) = Y_T(\lambda_2, s_2) - Y_T(\lambda_2, s_1) + Y_T(\lambda_1, s_1) - Y_T(\lambda_1, s_2)$.

As the process $Z_T(.,.)$ is not continuous with respect to its second coordinate, we introduce the process $L_T(.,.)$ with continuous sample path defined by

$$L_T(\lambda, s) = \frac{1}{\sqrt{T}} \int_0^\lambda l_T(\alpha, s) d\boldsymbol{\lambda}(\alpha) \tag{34}$$

where $l_T(.,.)$ is the polygonal line which joins the points $((\alpha, k/T), z_T(\alpha, k/T))$ where

$$z_T(\alpha, s) = \left| \sum_{j=1}^{[sT]} X_j h(\frac{j}{T}) e^{-ij\alpha} \right|^2 - \mathbf{E} \left| \sum_{j=1}^{[sT]} X_j h(\frac{j}{T}) e^{-ij\alpha} \right|^2 = \left| d_{[sT]}(\alpha) \right|^2 - \mathbf{E} \left| d_{[sT]}(\alpha) \right|^2.$$

We first prove that the continuous process $L_T(.,.)$ satisfies Csensov tightness criterion (Proposition 4), then to obtain the tightness of Z_T (see [11, Déf. 7.3.25-b p.219]) we prove the contiguity (see [**?**, Chap 3.1 pp.19-24] or [**?**, Chap. 6 pp.85-91]]) of $\widetilde{Z}_T$ and L_T (Proposition 5).

Proposition 4 *Assume that (X_j) is in $\mathcal{G}$, then the family of processes (L_T) is tight.*

Proof Since here, we have $L_T(0, 0) = L_T(0, .) = L_T(., 0) = 0$ the first three conditions of Csensov criterion are obvious. Define for $B = [\lambda_1, \lambda_2] \times [s_1, s_2]$

$$\widehat{L}_T(B) = \frac{1}{\sqrt{T}} \int_{\lambda_1}^{\lambda_2} [l_T(\alpha, s_2) - l_T(\alpha, s_1)] \, d\boldsymbol{\lambda}(\alpha). \tag{35}$$

Our aim is to control

$$\mathbf{E} \left| \widehat{L}_T(B) \right|^2 = \frac{1}{T} \mathbf{E} \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{\lambda_2} [l_T(\alpha, s_2) - l_T(\alpha, s_1)] \, [l_T(-\beta, s_2) - l_T(-\beta, s_1)] \, d\boldsymbol{\lambda}(\alpha) d\boldsymbol{\lambda}(\beta). \tag{36}$$

The technical part of this proof is to control $\mathbf{E} \left| \widehat{L}_T(B) \right|^2$ when $0 \leq s_2 - s_1 \leq \frac{1}{T}$. The general case is a direct consequence of this particular one. For this and before considering the general case, let us denote by

$$\Delta_n(\alpha) = \sum_{j=1}^n e^{-ij\alpha} \text{ the Dirichlet kernel and } \varphi_n(\lambda) = \int_0^\lambda \Delta_n(\alpha) d\boldsymbol{\lambda}(\alpha). \tag{37}$$

[39, Lemma 8.2 p.57] assure that the functions φ_n are continuous on the torus and are uniformly bounded in n and λ so one can find a uniform modulus of continuity $\omega(.)$ for the family $(\varphi_n)_{n \in \mathbf{N}}$ i.e. which satisfies $\sup_{0 < |\mu - \lambda| < \delta} |\varphi_n(\mu) - \varphi_n(\lambda)| \leq \omega(\delta)$ for all integer n, we prove the following lemma

Lemma 4.1 *Let $\omega(.)$ be an uniform modulus of continuity for all the φ_n defined by (37), then for $B = [\lambda_1, \lambda_2] \times [s_1, s_2]$ such that $[s_1 T] \leq s_1 T < s_2 T \leq [s_1 T] + 1$, the following inequality holds*

$$\mathbf{E} \left| \widehat{L}_T(B) \right|^2 \leq C_1 (s_2 - s_1) \left[(\lambda_2 - \lambda_1) \vee \omega^2(\lambda_2 - \lambda_1) \right].$$

Proof We note $k=[s_1T]$ and $Y_j=h(j/T)X_j$ the tapered data. Then, using that $0\le s_2-s_1\le 1/T$, it follows

$$l_T(\alpha,s_2)-l_T(\alpha,s_1)=T(s_2-s_1)\left(z_T(\alpha,\frac{k+1}{T})-z_T(\alpha,\frac{k}{T})\right) \tag{38}$$

and

$$\mathbf{E}\left|\widehat{L}_T(B)\right|^2\le(s_2-s_1)\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2}\mathbf{cov}\left[z_T(\alpha,\frac{k+1}{T})-z_T(\alpha,\frac{k}{T}),z_T(\beta,\frac{k+1}{T})-z_T(\beta,\frac{k}{T})\right]d\boldsymbol{\lambda}(\alpha)d\boldsymbol{\lambda}(\beta). \tag{39}$$

Now, noting that

$$z_T(\alpha,\frac{k+1}{T})-z_T(\alpha,\frac{k}{T})=Y_{k+1}^2-\mathbf{E}\,Y_{k+1}^2+2Y_{k+1}\mathbf{Re}\sum_{j=1}^{k}Y_je^{-i(k+1-j)\alpha}-2\mathbf{E}\left(Y_{k+1}\mathbf{Re}\sum_{j=1}^{k}Y_je^{-i(k+1-j)\alpha}\right),$$

we obtain

$$\mathbf{E}\left|\widehat{L}_T(B)\right|^2\le(s_2-s_1)(\lambda_2-\lambda_1)^2\mathbf{var}\ Y_{k+1}^2+\mathbf{Re}\,(R_1)+R_2,\ \text{with} \tag{40}$$

$$R_1=4(s_2-s_1)(\lambda_2-\lambda_1)\int_{\lambda_1}^{\lambda_2}\mathbf{cov}\left(Y_{k+1}\sum_{j=1}^{k}Y_je^{-i(k+1-j)\alpha},Y_{k+1}^2\right)d\boldsymbol{\lambda}(\alpha),$$

$$R_2=4(s_2-s_1)\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2}\mathbf{cov}\left(Y_{k+1}\sum_{j=1}^{k}Y_j\cos[(k+1-j)\alpha],Y_{k+1}\sum_{j=1}^{k}Y_j\cos[(k+1-j)\beta]\right)d\boldsymbol{\lambda}(\alpha)d\boldsymbol{\lambda}(\beta).$$

Since (X_j) is a Gaussian stationary process, $\mathbf{var}\ X_j^2$ is constant, and

$$(s_2-s_1)(\lambda_2-\lambda_1)^2\mathbf{var}\ Y_{k+1}^2\le\pi\mathbf{var}\ X_0^2\,||h||_\infty^2\,(s_2-s_1)(\lambda_2-\lambda_1). \tag{41}$$

The two last right terms of (40) are controlled by the two following lemma

Lemma 4.2 : $|R_1|\le 8\sqrt{\pi}\mathbf{E}\,X_0^2\,||h||_\infty^4\,||f||_2\,(s_2-s_1)(\lambda_2-\lambda_1)$

Proof Using the identity for Gaussian vector

$$\mathbf{cov}\left(Y_{k+1}\sum_{j=1}^{k}Y_je^{-i(k+1-j)\alpha},Y_{k+1}^2\right)=2\mathbf{E}\,Y_{k+1}^2\sum_{j=1}^{k}e^{-i(k+1-j)\alpha}\mathbf{E}\,Y_jY_{k+1}$$

$$=2\mathbf{E}\,X_0^2h(\frac{k+1}{T})^3\int_{\mathbf{T}}f(\beta)\sum_{j=1}^{k}h(\frac{j}{T})e^{-i(k+1-j)(\alpha+\beta)}d\boldsymbol{\lambda}(\beta)=2\mathbf{E}\,X_0^2h(\frac{k+1}{T})^3\widetilde{H}_1^k*f(\alpha) \tag{42}$$

with $\widetilde{H}_1^k(\alpha)=e^{-i(k+1)\alpha}\overline{H_1^k(\alpha)}$.

Lemma 4.3 *Let $\widetilde{H}_1^k(\alpha)=\sum_{j=1}^k h(j/T)e^{-ij(k+1-j)\alpha}$, then for all f in $L_2(\mathbf{T})$,* $\left|\left|\widetilde{H}_1^k*f\right|\right|_2\le||h||_\infty\,||f||_2$.

Proof Let $\widehat{f}(m)$ be the Fourier coefficients of f, then

$$\int_{\mathbf{T}}\widetilde{H}_1^k(\alpha)f(\alpha+\beta)d\boldsymbol{\lambda}(\alpha)=\int_{\mathbf{T}}\sum_{j=1}^{k}h(\frac{j}{T})e^{i(j-k-1)\alpha}\sum_{m\in\mathbf{Z}}\widehat{f}(m)e^{im(\alpha+\beta)}d\boldsymbol{\lambda}(\alpha)$$

$$=\sum_{m\in\mathbf{Z}}\widehat{f}(m)e^{im\beta}\int_{\mathbf{T}}\sum_{j=1}^{k}h(\frac{j}{T})e^{i(j+m-k-1)\alpha}d\boldsymbol{\lambda}(\alpha)=\sum_{j=1}^{k}\widehat{f}(k+1-j)h(\frac{j}{T})e^{i(k+1-j)\beta}.$$

By Parseval equality, it follows $\left|\left|\widetilde{H}_1^k * f\right|\right|_2^2 = \sum_{j=1}^{k} h^2(\frac{j}{T}) \left|\widehat{f}(k+1-j)\right|^2 \leq ||h||_\infty^2 \, ||f||_2^2$.

Q.E.D.

Now, applying Schwarz inequality and Lemma 4.3 to (42), it follows

$$|R_1| \leq 8\sqrt{\pi}\mathbf{E}\, X_0^2 \, ||h||_\infty^3 \, (s_2-s_1)(\lambda_2-\lambda_1) \left|\left|\widetilde{H}_1^k * f\right|\right|_2 \leq 8\sqrt{\pi}\mathbf{E}\, X_0^2 \, ||h||_\infty^4 \, ||f||_2 \, (s_2-s_1)(\lambda_2-\lambda_1).$$

Q.E.D.

Lemma 4.4 : $|R_2| \leq 4\,||h||_\infty^2 \,(||f||_2^2 + (\mathbf{E}\, X_0^2)^2)(s_2-s_1)\left[(\lambda_2-\lambda_1) \vee \omega^2(\lambda_2-\lambda_1)\right]$

Proof Using again the identity for Gaussian vector

$$\begin{aligned}
R_2 &= 4(s_2-s_1)\int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2} \mathbf{cov}\left(Y_{k+1}\sum_{j=1}^{k} Y_j \cos[(k+1-j)\alpha], Y_{k+1}\sum_{j=1}^{k} Y_j \cos[(k+1-j)\beta]\right) d\boldsymbol{\lambda}(\alpha) d\boldsymbol{\lambda}(\beta) \\
&= S_1 + S_2, \text{ with} \qquad (43)\\
S_1 &= 4(s_2-s_1)\left(\int_{\lambda_1}^{\lambda_2} \mathbf{E}\left[Y_{k+1}\sum_{j=1}^{k} Y_j \cos[(k+1-j)\alpha]\right] d\boldsymbol{\lambda}(\alpha)\right)^2, \\
S_2 &= 4(s_2-s_1)\mathbf{E}\, Y_{k+1}^2 \mathbf{E}\left[\int_{\lambda_1}^{\lambda_2}\sum_{j=1}^{k} Y_j \cos[(k+1-j)\alpha] d\boldsymbol{\lambda}(\alpha)\right]^2.
\end{aligned}$$

The bound for S_1 is close the one of R_1 developed in Lemma 4.2. By Schwarz inequality and lemma 4.3 it follows

$$\begin{aligned}
|S_1| &\leq 4(s_2-s_1)\left|\int_{\lambda_1}^{\lambda_2} \mathbf{E}\left[Y_{k+1}\sum_{j=1}^{k} Y_j e^{-i(k+1-j)\alpha}\right] d\boldsymbol{\lambda}(\alpha)\right|^2 \leq 4(s_2-s_1)\left|\int_{\lambda_1}^{\lambda_2} \widetilde{H}_1^k * f(\alpha) d\boldsymbol{\lambda}(\alpha)\right|^2 \\
&\leq 4(s_2-s_1)(\lambda_2-\lambda_1)\,||h||_\infty^2 \, ||f||_2^2. \qquad (44)
\end{aligned}$$

To control S_2 in (43), let us first consider ω a uniform modulus of continuity for all the integrated Dirichlet kernels $\int_0^\lambda \Delta_n(\alpha) d\boldsymbol{\lambda}(\alpha)$ where $\Delta_n = H_0^k$ (see 1), see for example [39, lemma Lemma 8.2 p.57] for the existence of such modulus.

Lemma 4.5 $|S_2| \leq 4\,||f||_2\,(\mathbf{V}\,(h))^2 \mathbf{E}\, Y_0^2 (s_2-s_1)\omega^2(\lambda_2-\lambda_1)$

Proof Let us denote by $\widetilde{Y}_j = Y_{k+1-j}$, we have to control the following term

$$V = \mathbf{E}\left(\int_{\lambda_1}^{\lambda_2}\sum_{j=1}^{k} \widetilde{Y}_j \cos(j\alpha) d\boldsymbol{\lambda}(\alpha)\right)^2 \leq \mathbf{E}\left|\int_{\lambda_1}^{\lambda_2}\sum_{j=1}^{k} \widetilde{Y}_j e^{-ij\alpha} d\boldsymbol{\lambda}(\alpha)\right|^2. \qquad (45)$$

Considering this last inequality, it follows

$$\begin{aligned}
V &\leq \mathbf{E}\left(\int_{\lambda_1}^{\lambda_2}\sum_{j_1=1}^{k} \widetilde{Y}_j e^{-ij_1\alpha} d\boldsymbol{\lambda}(\alpha) \int_{\lambda_1}^{\lambda_2}\sum_{j_2=1}^{k} \widetilde{Y}_j e^{ij_2\beta} d\boldsymbol{\lambda}(\beta)\right) = \int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2}\sum_{j_1=1}^{k}\sum_{j_2=1}^{k} \mathbf{E}\,(\widetilde{Y}_{j_1}\widetilde{Y}_{j_2}) e^{-ij_1\alpha} e^{ij_2\beta} d\boldsymbol{\lambda}(\alpha) d\boldsymbol{\lambda}(\beta) \\
&= \int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2}\sum_{j_1=1}^{k}\sum_{j_2=1}^{k}\left[h(\frac{k+1-j_1}{T})h(\frac{k+1-j_2}{T})\int_{\mathbf{T}} f(\gamma) e^{-i(j_2-j_1)\gamma} d\boldsymbol{\lambda}(\gamma)\right] e^{-ij_1\alpha} e^{ij_2\beta} d\boldsymbol{\lambda}(\alpha) d\boldsymbol{\lambda}(\beta)
\end{aligned}$$

$$= \int_{\mathbf{T}} \left[\int_{\lambda_1}^{\lambda_2} \sum_{j_1=1}^{k} h(\frac{k+1-j_1}{T}) e^{-ij_1(\alpha-\gamma)} d\boldsymbol{\lambda}(\alpha) \right] \left[\int_{\lambda_1}^{\lambda_2} \sum_{j_2=1}^{k} h(\frac{k+1-j_2}{T}) e^{ij_2(\beta-\gamma)} d\boldsymbol{\lambda}(\beta) \right] f(\gamma) d\boldsymbol{\lambda}(\gamma)$$

$$= \int_{\mathbf{T}} \left| \int_{\lambda_1}^{\lambda_2} \sum_{j=1}^{k} h(\frac{k+1-j}{T}) e^{-ij(\alpha-\gamma)} d\boldsymbol{\lambda}(\alpha) \right|^2 f(\gamma) d\boldsymbol{\lambda}(\gamma).$$

Let us denote $\widetilde{\lambda}_i = \lambda_i - \gamma$. By an Abel transformation ($e^{-ij\alpha} = \Delta_j(\alpha) - \Delta_{j-1}(\alpha)$), we have

$$\int_{\lambda_1}^{\lambda_2} \sum_{j=1}^{k} h(\frac{k+1-j}{T}) e^{-ij(\alpha-\gamma)} d\boldsymbol{\lambda}(\alpha) = h(\frac{1}{T}) \int_{\widetilde{\lambda}_1}^{\widetilde{\lambda}_2} \Delta_k(\alpha) d\boldsymbol{\lambda}(\alpha) + \sum_{j=1}^{k-1} \left[h(\frac{k+1-j}{T}) - h(\frac{k-j}{T}) \right] \int_{\widetilde{\lambda}_1}^{\widetilde{\lambda}_2} \Delta_j(\alpha) d\boldsymbol{\lambda}(\alpha)$$

it follows that

$$\left| \int_{\lambda_1}^{\lambda_2} \sum_{j=1}^{k} h(\frac{k+1-j}{T}) e^{-ij(\alpha-\gamma)} d\boldsymbol{\lambda}(\alpha) \right| \leq \mathbf{V}(h).\omega(\lambda_2 - \lambda_1)$$

and using Schwarz inequality

$$V = \mathbf{E} \left(\int_{\lambda_1}^{\lambda_2} \sum_{j=1}^{k} \widetilde{Y}_j \cos(j\alpha) d\boldsymbol{\lambda}(\alpha) \right)^2 \leq \sqrt{2\pi}\, ||f||_2\, (\mathbf{V}(h))^2 \omega^2(\lambda_2 - \lambda_1).$$

Reporting this last inequality in the definition of S_2 (see (43)), it follows that

$$|S_2| \leq 4\, ||f||_2\, ||h||_\infty^2\, (\mathbf{V}(h))^2 \mathbf{E}\, X_0^2 (s_2 - s_1) \omega^2(\lambda_2 - \lambda_1).$$

Q.E.D.

Now, using (44) and Lemma 4.5, we obtain for R_2 defined in (43)

$$|R_2| \leq C_2(s_2 - s_1) \left(\omega^2(\lambda_2 - \lambda_1) \vee (\lambda_2 - \lambda_1) \right)$$

so the Lemma 4.4 is proved

Q.E.D.

Finally using (41), Lemma 4.2 and Lemma 4.4, we prove that when $[s_1 T] \leq s_1 T < s_2 T \leq [s_1 T] + 1$, the following inequality holds

$$\mathbf{E} \left| \widehat{L}_T(B) \right|^2 \leq C_1(s_2 - s_1) \left[(\lambda_2 - \lambda_1) \vee \omega^2(\lambda_2 - \lambda_1) \right],$$

this ends the proof of Lemma 4.1

Q.E.D.

To close the proof of the tightness, we have to extend the result of Lemma 4.1 to the general case where $|s_2 - s_1| \geq 1/T$. It is the object of the following Lemma.

Lemma 4.6 *Let $\omega(.)$ an uniform modulus of continuity for all φ_n (see (37)), then for $B = [\lambda_1, \lambda_2] \times [s_1, s_2]$ such that $|s_2 - s_1| \geq 1/T$ the following inequality holds*

$$\mathbf{E} \left| \widehat{L}_T(B) \right|^2 \leq C_2(s_2 - s_1) \left[(\lambda_2 - \lambda_1) \vee \omega^2(\lambda_2 - \lambda_1) \right].$$

Proof We consider $t_i = \frac{[s_1 T] + i}{T}$, for $i \geq 0$ and we note $p = [s_2 T] - [s_1 T]$. So we have

$$t_0 \leq s_1 < t_1 < \ldots < t_p \leq s_2 < t_{p+1}.$$

Clearly,

$$\widehat{L}_T(B) = \widehat{L}_T([\lambda_1,\lambda_2]\times[s_1,t_1]) + \sum_{j=1}^{p-1}\widehat{L}_T([\lambda_1,\lambda_2]\times[t_j,t_{j+1}]) + \widehat{L}_T([\lambda_1,\lambda_2]\times[t_p,s_2]) \tag{46}$$

and by Schwarz inequality it follows, using that $1/T \le s_2 - s_1$ and $[s_2T]-[s_1T]-1 \le (s_2-s_1)T$

$$\begin{aligned}\mathbf{E}\left|\widehat{L}_T(B)\right|^2 &\le C_1(p+1)\frac{1}{T}\left[\omega^2(\lambda_2-\lambda_1)\vee(\lambda_2-\lambda_1)\right] \le C_1\frac{[s_2T]-[s_1T]+1}{T}\left[\omega^2(\lambda_2-\lambda_1)\vee(\lambda_2-\lambda_1)\right] \\ &\le 3C_1(s_2-s_1)\left[\omega^2(\lambda_2-\lambda_1)\vee(\lambda_2-\lambda_1)\right].\end{aligned}$$

Q.E.D.

As for $\delta>0$, $\tilde{\omega}(\delta)=\omega^2(\delta)\vee\delta$ clearly define a modulus of continuity, the fourth condition of Csensov criterion holds for $\mathbf{E}\,|\widehat{L}_T(B)|^2$ with $\gamma_1=2$ and $\gamma_2=1$ so the continuous process $(L_T)_{T\in\mathbf{N}}$ is tight in $\mathcal{C}([0,\pi]\times[0,1],\mathbf{R})$. So the proof of Proposition 4 is complete.

Q.E.D.

Proposition 5 *The two families of processes $(L_T)_T$ and $(Z_T)_T$ are contiguous (see [**?**, Chap 3.1 pp.19-24] or [**?**, Chap. 6 pp.85-91] for definition of the contiguity).*

Proof Since Proposition 1 holds we prove that (L_T) and the centered process $(\widetilde{Z}_T)$ (see (27)) are contiguous. For this we prove the following more general result:

$$P\left(\sup_{\lambda\in[0,\pi]}\sup_{s\in[0,1]}\left|L_T(\lambda,s)-\widetilde{Z}_T(\lambda,s)\right|>\epsilon\right)\to 0 \text{ as } T\to+\infty.$$

Applying the fourth condition of Csensov criterion to the process (L_T) with $B=[0,\lambda]\times[\frac{k}{T},\frac{k+1}{T}]$, it follows for $k=1,\ldots,T$

$$\mathbf{E}\left|\widehat{L}_T([0,\lambda]\times[\frac{k}{T},\frac{k+1}{T}])\right|^2 \le C\frac{\tilde{\omega}(\pi)}{T}. \tag{47}$$

As $\lambda\mapsto\widehat{L}_T([0,\lambda]\times[\frac{k}{T},\frac{k+1}{T}])$ is continuous on $[0,\pi]$, it exists λ_k such that

$$\sup_{\lambda\in[0,\pi]}\left|\widehat{L}_T([0,\lambda]\times[\frac{k}{T},\frac{k+1}{T}])\right| = \left|\widehat{L}_T([0,\lambda_k]\times[\frac{k}{T},\frac{k+1}{T}])\right|.$$

As k takes only a finite number of values, there exists k_0 such that

$$\sup_{k=1,\ldots,T}\left|\widehat{L}_T([0,\lambda_k]\times[\frac{k}{T},\frac{k+1}{T}])\right| = \sup_{k=1,\ldots,T}\sup_{\lambda\in[0,\pi]}\left|\widehat{L}_T([0,\lambda]\times[\frac{k}{T},\frac{k+1}{T}])\right| = \left|\widehat{L}_T([0,\lambda_{k_0}]\times[\frac{k_0}{T},\frac{k_0+1}{T}])\right|.$$

It follows from this last equality and from (47) that

$$\mathbf{E}\left(\sup_{k=1,\ldots,T}\sup_{\lambda\in[0,\pi]}\left|\widehat{L}_T([0,\lambda]\times[\frac{k}{T},\frac{k+1}{T}])\right|\right)^2 = \mathbf{E}\left|\widehat{L}_T([0,\lambda_{k_0}]\times[\frac{k_0}{T},\frac{k_0+1}{T}])\right|^2 \le C\frac{\tilde{\omega}(\pi)}{T}. \tag{48}$$

Now, as $\widetilde{Z}_T(\lambda,s)=L_T(\lambda,[sT]/T)$, it follows, using that l_T is piecewiese affine

$$L_T(\lambda,s)-\widetilde{Z}_T(\lambda,s) = L_T(\lambda,s)-L_T(\lambda,\frac{[sT]}{T}) = T(s-\frac{[sT]}{T})\widehat{L}_T([0,\lambda]\times[\frac{[sT]}{T},\frac{[sT]+1}{T}]).$$

Applying (48), its follows

$$\mathbf{E}\left(\sup_{\lambda\in[0,\pi]}\sup_{s\in[0,1]}\left|L_T(\lambda,s)-\widetilde{Z}_T(\lambda,s)\right|\right)^2 \le C\frac{\tilde{\omega}(\pi)}{T}.$$

Using Tchebicev inequality, we deduce that

$$P\left(\sup_{\lambda\in[0,\pi]}\sup_{s\in[0,1]}\left|L_T(\lambda,s)-\widetilde{Z}_T(\lambda,s)\right|>\epsilon\right)\le C\frac{\tilde{\omega}(\pi)}{T\epsilon^2}\to 0 \text{ as } T\to+\infty.$$

So (L_T) and (Z_T) are contiguous.

Q.E.D.

5 Asymptotic distribution of the test statistics under the null hypothesis

We prove in this section our main theorem that is Theorem 1. The convergence of the statistic constructed on (Y_1^T) is a direct consequence of relation (12) and Theorem 2: as the limiting process Z is in $\mathcal{C}([0,\pi]\times[0,1],\mathbf{R})$, we just need to apply the classical functional theorems in $\mathcal{C}$ (see [3, Th. 4.4 p 27 & Th. 5.1 p. 30]) and to remark that the sequence $(\frac{H_2^T}{T})_T$ converges to H_2 which is assumed to be finite and non zero.

We also have to prove that the two processes $(Y_j^T)_T$, $j=1,2$ have the same limiting distribution. For this we prove the convergence in distribution to 0 under the null hypothesis of the sequence of processes $(\Omega_T)_{T>0}$ defined by (13) and (14).

Theorem 3 *If $X_.$ is a process in $\mathcal{G}$ then $\Omega_T\to 0$ as $T\to\infty$.*

Proof Let us first investigate the finite distributions of $(\Omega_T(.,.))_T$

Lemma 5.1 *If $X_.$ is a process in $\mathcal{G}$ then process $(\Omega_T(.,.))_T$ converges in probability to 0.*

Proof The proof splits into to steps
Step 1 Process $(\Omega_T(.,.))_T$ is uniformly asymptotically unbiased. Remark that

$$\Omega_T(\lambda,s)=Z_T(\lambda,1)-Z_T(\lambda,s)-Z_T^*(\lambda,1-s) \tag{49}$$

where

$$Z_T^*(\lambda,1-s)=\frac{\check{H}_2^{T-[sT]}}{\sqrt{T}}\left(\check{F}_{T-[sT]}(\lambda)-F(\lambda)\right)$$

is the "reverse-time" process of Z_T constructed on $h(\frac{T}{T})X_T,\ldots,h(\frac{[sT]+1}{T})X_{[sT]+1}$. Clearly, the process Z_T^* has the same properties as Z_T. In particular, Z_T^* satisfies Proposition 1. Therefore, the same is true for Ω_T and we have $\sup_{(\lambda,s)\in[0,\pi]\times[0,1]}|\mathbf{E}\,\Omega_T(\lambda,s)|=o(1)$.

Step 2 For all (λ,s), the variance of $\Omega_T(\lambda,s)$ tends to 0 as $T\to\infty$. Following the same lines of proof as in Proposition 2 and considering that $Z_T^*(\lambda,1-s)$ is defined on the variables $X_T,\ldots,X_{[sT]+1}$ and using the taper h on the set $\{\frac{T}{T},\ldots,\frac{[sT]+1}{T}\}$ which converge to $[s,1]$, we prove that

$$\lim_{T\to\infty}\mathbf{cov}(Z_T^*(\lambda,1-s),Z_T^*(\lambda,1-s'))=\int_0^\lambda f^2(\alpha)d\boldsymbol{\lambda}(\alpha)\int_{[s,1]\wedge[s',1]}h^4(u)du=\int_0^\lambda f^2(\alpha)d\boldsymbol{\lambda}(\alpha)\int_{s\vee s'}^1 h^4(u)du.$$

Now, as $Z_T(\lambda,1)=Z_T^*(\lambda,1)$, using (49) to compute $\mathbf{var}\,\Omega_T(\lambda,s)$ gives

$$\lim_{T\to\infty}\mathbf{var}\,\Omega_T(\lambda,s)=2\lim_{T\to\infty}\mathbf{cov}(Z_T(\lambda,s),Z_T^*(\lambda,1-s)).$$

We just need to prove that $\mathbf{cov}(Z_T(\lambda,s),Z_T^*(\lambda,1-s))\to 0$. Following the same lines as in Lemma 2, it follows that

$$\mathbf{cov}(Z_T(\lambda,s),Z_T^*(\lambda,1-s))=\Psi_T^s*\widetilde{g}(0,0,0) \tag{50}$$

with $\widetilde{g}$ defined by (30). For all $\gamma = (\gamma_1, \gamma_2, \gamma_3)$

$$\Psi_T^s(\gamma) = \frac{1}{T} H_1^{[sT]}(\gamma_1) H_1^{[sT]}(\gamma_2) \check{H}_1^{T-[sT]}(\gamma_3) \check{H}_1^{T-[sT]}(-\gamma_1 - \gamma_2 - \gamma_3)$$

with $\check{H}_1^{T-k}(\alpha) = H_1^T(\alpha) - H_1^k(\alpha)$. Since $\Psi_T^s = \frac{H_4^{[sT]}}{T}(\Phi_T^{s,s,1,1} - \Phi_T^{s,s,1,s} - \Phi_T^{s,s,s,1} + \Phi_T^{s,s,s,s})$ (see (31) for the definition of $\Phi_T^{s_1,s_2,s_3,s_4}$) using that $\Phi_T^{s,s,1,1}$, $\Phi_T^{s,s,1,s}$, $\Phi_T^{s,s,s,1}$ and $\Phi_T^{s,s,s,s}$ are approximate identities (see Lemma 3.2) their product of convolution with $\tilde{g}$ at point $(0,0,0)$ tends to $\tilde{g}(0,0,0)$ as T tends to ∞. It follows that the right term of (50) tends to 0.

Applying Chebishev inequality, Lemma 5.1 is obtained.

Q.E.D.

Lemma 5.2 *If $X_.$ is a process in $\mathcal{G}$ then the sequence of processes $(\Omega_T)_{T\in\mathbf{N}}$ is tight.*

Proof Using that process $(\Omega_T(.,.))_T$ is uniformly asymptotically unbiased, it is enough to prove that $(\Omega_T - \mathbf{E}\,\Omega_T)$ is contiguous to the $\mathcal{C}$-tighted process $L'_T = \frac{1}{\sqrt{T}} \int_0^\lambda l'_T(\alpha,s) d\boldsymbol{\lambda}(\alpha)$ where $l'_T(\alpha,s)$ is the polygonal line which joins the points $((\alpha, k/T), z'_T(\alpha, k/T))$ with

$$z'_T(\alpha,s) = \left(\sum_{j_1=1}^{k} Y_j e^{-ij_1\alpha} \sum_{j_2=k+1}^{T} Y_j e^{ij_2\alpha}\right) - \mathbf{E}\left(\sum_{j_1=1}^{k} Y_j e^{-ij_1\alpha} \sum_{j_2=k+1}^{T} Y_j e^{ij_2\alpha}\right),$$

and with $Y_j = h(j/T)X_j$ and $k = [sT]$. Equation (35) to (39) are still valid, replacing z_T, l_T, L_T, $\widehat{L_T}$ by z'_T, l'_T, L'_T, $\widehat{L'_T}$, so that

$$\begin{aligned}
z'_T(\alpha, \frac{k+1}{T}) - z'_T(\alpha, \frac{k}{T}) &= \left(\sum_{j_1=1}^{k+1} Y_j e^{-ij_1\alpha} \sum_{j_2=k+2}^{T} Y_j e^{ij_2\alpha} - \sum_{j_1=1}^{k} Y_j e^{-ij_1\alpha} \sum_{j_2=k+1}^{T} Y_j e^{ij_2\alpha}\right) \\
&\quad -\mathbf{E}\left(\sum_{j_1=1}^{k+1} Y_j e^{-ij_1\alpha} \sum_{j_2=k+2}^{T} Y_j e^{ij_2\alpha} - \sum_{j_1=1}^{k} Y_j e^{-ij_1\alpha} \sum_{j_2=k+1}^{T} Y_j e^{ij_2\alpha}\right) \\
&= \left(Y_{k+1} e^{-i(k+1)\alpha} \sum_{j_2=k+2}^{T} Y_j e^{ij_2\alpha} - \sum_{j_1=1}^{k} Y_j e^{-ij_1\alpha} Y_{k+1} e^{i(k+1)\alpha}\right) \\
&\quad -\mathbf{E}\left(Y_{k+1} e^{-i(k+1)\alpha} \sum_{j_2=k+2}^{T} Y_j e^{ij_2\alpha} - \sum_{j_1=1}^{k} Y_j e^{-ij_1\alpha} Y_{k+1} e^{i(k+1)\alpha}\right) \\
&= \left[Y_{k+1}(\overline{d_k(\alpha)} - \check{d}_{T-k-1}(\alpha))\right] - \mathbf{E}\left[Y_{k+1}(\overline{d_k(\alpha)} - \check{d}_{T-k-1}(\alpha))\right]
\end{aligned}$$

with $d_k(\alpha)$ defined in (2) and $\check{d}_{T-k}(\alpha) = d_T(\alpha) - d_k(\alpha)$. It follows that, for $[s_1T] \le s_1T < s_2T \le [s_1T]+1$,

$$\mathbf{E}\,|\widehat{L'_T}(B)|^2 \le (s_2 - s_1) \int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2} \mathbf{cov}\left(Y_{k+1}(\overline{d_k(\alpha)} - \check{d}_{T-k-1}(\alpha)), Y_{k+1}(\overline{d_k(\beta)} - \check{d}_{T-k-1}(\beta))\right) d\boldsymbol{\lambda}(\alpha) d\boldsymbol{\lambda}(\beta).$$

The righthand term is similar to the quantity R_2 appearing in (40), therefore, it is controlled in a same way (see Lemma 4.4). Such control leads to the following inequality

$$\mathbf{E}\,|\widehat{L'_T}(B)|^2 \le C\,(s_2 - s_1)\left[(\lambda_2 - \lambda_1) \vee \omega^2(\lambda_2 - \lambda_1)\right]$$

which holds for $[s_1T] \leq s_1T < s_2T \leq [s_1T]+1$. Now, as (46)-(48) hold for $\widehat{L'_T}(B)$, Lemma 4.6 and Proposition 5 can be extended to process $\widehat{L'_T}(B)$ and Ω_T. Therefore $(\Omega_T)_{T>0}$ is contiguous to a $\mathcal{C}$-tighted process so is tighted (see [3]).

Q.E.D.

This proves Theorem 3 together with the asymptotic equivalence of the sequence $(Y_T^1)_{T>0}$ and $(Y_T^2)_{T>0}$. So the two families of test statistics $S_T^1(h)$ and $S_T^2(h)$ are also asymptotically equivalent

Q.E.D.

6 Critical region and applications

First, our purpose here, is to obtain the asymptotic form of the reject region. Then, we prove the consistency of our tests when G_2 is known or not. Finally, we present some numerical simulation results. We observe on these simulations that tapering improves detection.

6.1 Critical region

We establish here the following result

Corollary 2 *When T is large, the level of the test associated with the critical regions R_T^j, $j=1,2$ can be approximated using the limits*

$$\lim_{T\to+\infty} \mathbf{P}_{\mathcal{H}_0}(R_T^j) = \mathbf{P}(\sup_{u\in[0,1]}\sup_{s\in[0,1]}\left|B(u,s) - \frac{H_2[H_4^{-1}(s.H_4)]}{H_2}B(u,1)\right| > c)$$

where $H_4^{-1}(.)$ is the inverse or pseudo-inverse function of $H_4(.)$ and where B is the Gaussian process of $\mathcal{C}([0,1]^2,\mathbf{R})$ with mean zero and covariance function

$$\mathbf{E}\,B(u,s)B(u',s') = u\wedge u'.s\wedge s'$$

Proof Since the two statistics $S_T^j(h)$, $j=1,2$ have the same limit under $\mathcal{H}_0$ we only prove Corollary1.1 for $S_T^1(h)$. Let us consider the time change

$$u = \frac{G_2(\lambda)}{G_2} \qquad u' = \frac{G_2(\lambda')}{G_2} \qquad v = \frac{H_4(s)}{H_4} \qquad v' = \frac{H_4(s')}{H_4}$$

with $G_2(\lambda) = \int_0^\lambda f^2(\alpha)d\boldsymbol{\lambda}(\alpha)$ and $H_4(.)$ defined in (1). Denote by $\tilde{Z}$ the process defined on $[0,1]\times[0,1]$ by $\tilde{Z}(u,v) = Z(\lambda,s)$. It is easy to verify

$$\sup_{\lambda\in[0,\pi]}\sup_{s\in[0,1]}\left|\frac{1}{H_2}\left(Z(\lambda,s) - \frac{H_2(s)}{H_2}Z(\lambda,1)\right)\right| = \frac{1}{H_2}\sup_{u\in[0,1]}\sup_{v\in[0,1]}\left|\widetilde{Z}(u,v) - \frac{H_2(H_4^{-1}(H_4.v))}{H_2}\widetilde{Z}(u,1)\right|.$$

The process $\tilde{Z}/\sqrt{G_2H_4}$ has the same covariance function that the process B defined in Corollary 1.1. So, it is enough to prove Corollary 1.1

Q.E.D.

6.2 Consistency and practical use

If G_2 is known, it is clear that under $\mathcal{H}_1$ the statistic $S_T^2(h)$ converges in probability to $+\infty$. So our test is consistent. To get such a result for $S_T^1(h)$, we need more assumptions on the two processes before and after the change-point, due to the presence of the crossed term Ω_T (see (13) and (14)), for example.

Theorem 4 *Under $\mathcal{H}_1$, if the processes $\widetilde{X}^1$ and $\widetilde{X}^2$ are independent, the statistic $S_T^1(h)$ converges in probability to $+\infty$.*

Proof We prove that $S_T^1(h)$ has the same limit as $S_T^2(h)$. Using (13), we only have to prove that the process Ω_T is bounded in probability. For this, we remark first that, as the processes $\widetilde{X}^1$ and $\widetilde{X}^2$ being independent, Ω_T is unbiased. Let $\widetilde{\Phi}_T^{s_0} = \Phi_T^{s_0,s_0,s_0,s_0}$ defined in (31) and

$$\widehat{\Phi}_T^{s_0}(\gamma) = \frac{K_1^{[s_0T]}(\gamma_1)K_1^{[s_0T]}(\gamma_2)K_1^{[s_0T]}(\gamma_3)K_1^{[s_0T]}(-\sum_{j=1}^{3}\gamma_j)}{H_2^T - H_2^{[s_0T]}}$$

with $K_1^{[sT]}(\alpha) = \sum_{j=[sT]+1}^{T} h(\frac{j}{T})e^{-ij\alpha}$. Following the proof of Proposition 2 we obtain

$$\begin{aligned}\mathbf{var}^2\Omega_T(\lambda_0, s_0) \quad \leq \quad & \frac{H_2^{[s_0T]}}{T}\int_{\mathbf{T}^3}\widetilde{\Phi}_T^{s_0}(\gamma)d\boldsymbol{\lambda}(\gamma)\int_{\mathbf{T}} f_1(\gamma_1-\alpha)f_1(\gamma_2+\alpha)d\boldsymbol{\lambda}(\alpha)\\ & +\frac{H_2^T - H_2^{[s_0T]}}{T}\int_{\mathbf{T}^3}\widehat{\Phi}_T^{s_0}(\gamma)d\boldsymbol{\lambda}(\gamma)\int_{\mathbf{T}} f_2(\gamma_1-\alpha)f_2(\gamma_2+\alpha)d\boldsymbol{\lambda}(\alpha) \qquad (51)\end{aligned}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ with $d\boldsymbol{\lambda}(\gamma) = d\boldsymbol{\lambda}(\gamma)_1 d\boldsymbol{\lambda}(\gamma)_2 d\boldsymbol{\lambda}(\gamma)_3$, f_1 and f_2 denote the spectral density before and after change-point. The sequence of kernels $(\widetilde{\Phi}_T^{s_0})_T$ is an approximated identity for convolution (see Lemma 3.2). It is easy to see that the sequence $(\widehat{\Phi}_T^{s_0})_T$ defined also an approximated identity for convolution. So the right term in (51) converges and the covariance is bounded. Applying the Bienayme-Chebyshev inequality, we get the announced result.

Q.E.D.

Consider now the case where G_2 is unknown. We also have under $\mathcal{H}_1$ to control the estimated $\widetilde{G}_2$ defined in (8) which appears in the critical regions (see (7). In fact, we can show that it converges under $\mathcal{H}_1$ to a finite value which is

$$\frac{1}{2}\left[s_0^2\|f_1\|_2^2 + (1-s_0)^2\|f_2\|_2^2\right]^{1/2}.$$

So the test can be extended to the case G_2 unknown.

6.3 Numerical results

To present our numerical results we consider the polynomial taper

$$h_{\rho,n}(x) = \begin{cases} 4^n.(\frac{x}{\rho})^n.(1-\frac{x}{\rho})^n & x \in [0, \frac{\rho}{2}),\\ 1 & x \in [\frac{\rho}{2}, 1-\frac{\rho}{2}],\\ h_{\rho,n}(1-x) & x \in (1-\frac{\rho}{2}, 1],\end{cases}$$

with $n \geq 1$ and $0 \leq \rho \leq 1$. The parameter ρ controls how much tapered the data are: as ρ goes to 0, $h_{\rho,n}$ tends to 1 which corresponds to the untapered case. The parameter n controls the smoothness of the tapering: the larger n is, the smoother the tapering.

Before presenting some simulation results, we have to say a few words about the tabulation of the limiting distribution.

Since the limit distribution of our test is not independent from h, we need to tabulate it for each h. It may be of interest to bypass this problem by looking for a family of tapers $(h_T)_T$ indexed by T such that h_T converges to 1 when T goes to $+\infty$. Doing this we obtain the following result

Theorem 5 *If $(h_{\rho_T,n_T})_T$ is a sequence of tapers such that*

- *$(\rho_T)_T$ is a sequence of integer which tends to 0 when T goes to $+\infty$,*
- *$\liminf_{T\to\infty} T.\rho_T > 0$,*

- $(n_T)_T$ *is a bounded sequence such that* $n_T \geq 1$,

the level of the test associated to the critical regions R_T^j, $j = 1, 2$ *can be approximated using the following limits*

$$\lim_{T\to+\infty} \mathbf{P}_{\mathcal{H}_0}(R_T^j) = \mathbf{P}(\sup_{u\in[0,1]} \sup_{s\in[0,1]} |W(u,s)| > c)$$

where $W(u,s) = B(u,s) - s\,B(u,1)$, *and where* B *is defined in corollary 1.1.*

Proof The proof may be found in totality in the thesis [33, Th 10.0.12 pp. 90-94].

Q.E.D.

In fact for small T, it is more interesting to adapt our taper $h_{\rho,n}$ to the expecting number of false alarms. The following tables present for $T = 50$ and $T = 150$ the percentage of false alarms detected with each taper for 1000 trajectories of *i.i.d.* $N(0,1)$

- $T = 50$

α	30%	20%	15%	10%	5%	2.5%	1%
$h \equiv 1$	8	5	2	2	0	0	0
$h = h_{0.10,4}$	8	4	4	2	1	0	0
$h = h_{0.20,4}$	6	4	4	1	1	0	0
$h = h_{0.50,4}$	11	5	3	1	0	0	0
$h = h_{0.60,4}$	10	4	2	1	0	0	0
$h = h_{0.70,4}$	14	8	7	3	1	0	0
$h = h_{0.80,4}$	19	12	10	8	3	1	0
$h = h_{0.85,4}$	21	14	12	**11**	**7**	**3**	0
$h = h_{0.90,4}$	**29**	**19**	**15**	**11**	11	6	3

- $T = 150$

α	30%	20%	15%	10%	5%	2.5%	1%
$h \equiv 1$	15	8	5	3	1	1	0
$h = h_{0.10,4}$	22	16	13	8	6	0	0
$h = h_{0.20,4}$	25	18	13	7	3	1	0
$h = h_{0.50,4}$	28	16	13	5	1	1	0
$h = h_{0.55,4}$	26	15	12	5	1	1	0
$h = h_{0.60,4}$	25	**17**	**15**	**10**	**4**	1	0
$h = h_{0.65,4}$	**33**	**22**	17	13	**4**	2	0

On the first table, we see that the $h_{\rho,n}$ adapted taper is, for $T = 50$, obtained for ρ between 0.85 and 0.90. For $T = 150$, the second table shows that we have to take a value of ρ between 0.60 and 0.65. As the value of n does not change these results significantly we have only worked with $n = 4$.

For these two adapted tapers, we have tested a change-point in a process X_t constructed as follows

- $X_0, \ldots, X_{[s_0T]}$ are *i.i.d.* Gaussian $N(0,1)$.
- $X_{[s_0T]+1}, \ldots, X_T$ are the trace of an $AR(1)$ with root 0.3.

The following table gives the percentage of alarms detected when $s = 0.1$ and $s = 0.5$ and $s = 0.9$ with or without taper

- $T = 50$

	α	30%	20%	15%	10%	5%	2.5%	1%
$s_0 = 0.1$	$h \equiv 1$	24	25	24	23	20	16	15
	$h = h_{0.9,4}$	44	39	37	36	32	30	26
$s_0 = 0.5$	$h \equiv 1$	14	14	9	7	7	4	3
	$h = h_{0.9,4}$	26	21	18	14	13	10	6
$s_0 = 0.9$	$h \equiv 1$	9	6	3	3	3	1	1
	$h = h_{0.9,4}$	32	20	13	11	6	6	0

- $T = 150$

	α	30%	20%	15%	10%	5%	2.5%	1%
$s_0 = 0.1$	$h \equiv 1$	34	23	20	15	11	7	5
	$h = h_{0.6,4}$	40	31	23	22	19	16	9
$s_0 = 0.5$	$h \equiv 1$	28	19	17	16	12	11	9
	$h = h_{0.6,4}$	32	21	18	13	11	10	6
$s_0 = 0.9$	$h \equiv 1$	14	11	8	6	3	3	0
	$h = h_{0.6,4}$	23	12	9	7	5	2	1

These last two tables clearly show the effect of tapering when T is small ($T = 50$). In this case, we have, on average, twice more alarms detected with the taper. The case $T = 150$ shows that, even if we always have a better score with the taper, this taper effect is less when T grows up.

To conclude this numerical part we represent the field $Y_T^2(.,.)$ with or without taper. We clearly see on the following two figures what the taper effect is: smoothness and concentration.

fig 1 : Y_T^2 with tapering

fig 2 : Y_T^2 without tapering

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